Journal of Applied Mechanics and Technical Physics, Vol. 46, No. 2, pp. 191–201, 2005

EXACT SOLUTIONS OF LINEARIZED EQUATIONS OF CONVECTION OF A WEAKLY COMPRESSIBLE FLUID

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UDC 532.517.013.4:536.252

A mathematical model of fluid convection under microgravity conditions is considered. The equation of state is used in a form that allows considering the fluid as a weakly compressible medium. Based on the previously proposed mathematical model of convection of a weakly compressible fluid, unsteady convective motion in a vertical band, with a heat flux periodic in time set on the solid boundaries of this band, is considered. This model of convection allows one to study the problem with the boundary thermal model oscillating in an antiphase rather than in-phase mode, while the latter was required for the model of microconvection of an isothermally incompressible fluid. Exact solutions for velocity components and temperature are derived, and the trajectories of fluid particles are constructed. For comparison, the trajectories predicted by the classical Oberbeck–Boussinesq model of convection and by the microconvection model are presented.

Key words: *convection, weakly compressible fluid, periodic heat flux.*

1. Formulation of the Problem. This work continues the study of convective motion of a heat-conducting fluid. Various aspects of mathematical simulation and rigorous mathematical substantiation of convection models are described in famous monographs [1–4]. The classical equations of convection are the Oberbeck–Boussinesq equations. To study convection under microgravity conditions and at the microscopic scale, we use the model of microconvection of an isothermally incompressible fluid proposed by Pukhnachov (see [1, 2, 5]). Allowance for the non-solenoidal nature of the velocity field leads to non-Boussinesq effects in fluid flows, which are especially well manifested in considering unsteady problems [6, 7].

In studying convection in closed domains with solid impermeable boundaries, it was noted that the system of equations of microconvection of an isothermally incompressible fluid admits a correct formulation of the initial-boundary problem only for the heat flux set at the boundary and a zero total heat flux, which is a necessary condition for solving the problem. Pukhnachov [5] proposed a model of convection of a weakly compressible fluid, which is free from this rigorous condition. The viscosity ν and thermal diffusivity χ are assumed to be constant, and the equation of state is used in the following form:

$$\rho = (1 + \delta p)/(1 + \varepsilon T). \tag{1}$$

Equation (1) is written in a dimensionless form. Here p and T are the deviations of pressure and temperature from certain equilibrium values p_0 and T_0, l is the characteristic length scale, $v_* = \chi/l$ is the velocity, $t_* = l/v_* = l^2/\chi$ is the time, $p_* = \rho_0\nu\chi/l^2$ is the pressure, T_* is the temperature, and ρ_0 is the density. Two basic small dimensionless parameters that appear in the problem are $\delta = \gamma p_*$ (parameter of compressibility) and $\varepsilon = \beta T_*$ (Boussinesq number). Indeed, the Boussinesq number is a quantity of the order of 10^{-5}–10^{-3} because the temperature coefficient of volume expansion β is small even if the temperature difference is large (e.g., 50 K). The parameter δ proportional to the isothermal compressibility coefficient γ is a quantity of the order of 10^{-14}– 10^{-9}, because we have $\gamma \in [10^{-10}, 10^{-9}]$ for conventional fluids (see [3, 5]).

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The system of hydrodynamic equations in the dimensionless form with allowance for Eq. (1) is written as follows [5]:

$$\frac{1+\delta p}{1+\varepsilon T}\,\frac{d\boldsymbol{V}}{dt} = \mathrm{Pr}\,[\nabla(-p+\bar{\xi}{-}er\boldsymbol{V}) + \Delta\boldsymbol{V}] + \frac{\eta \mathrm{Pr}\,(1+\delta p)}{1+\varepsilon T}\,\boldsymbol{g}_0; \tag{2}$$

$$\frac{\delta}{1+\delta p}\,\frac{dp}{dt} - \frac{\varepsilon}{1+\varepsilon T}\,\frac{dT}{dt} + {-er}\boldsymbol{V} = 0; \tag{3}$$

$$\frac{1+\delta p}{1+\varepsilon T}\,\frac{dT}{dt} - \frac{\epsilon_2+\varepsilon\epsilon_1 T}{1+\varepsilon T}\,\frac{dp}{dt} = \Delta T + \epsilon_1\Phi. \tag{4}$$

Here $\mathrm{Pr} = \nu/\chi$ is the Prandtl number, $\eta = gl^3/(\nu\chi)$ is the microconvection parameter, $\epsilon_1 = \nu v_*/(lc_pT_*) = \nu\chi/(l^2c_pT_*)$, $\epsilon_2 = \varepsilon\epsilon_1T_0/T_* = \beta\nu\chi T_0/(l^2c_pT_*)$, c_p is the heat capacity of the fluid at constant pressure, $\bar{\xi} = 1+\xi$ [$\xi = \lambda/(\rho_0\nu)$ is the ratio of the coefficients of the second and first viscosity], $\boldsymbol{g}_0 = \boldsymbol{g}/g$ ($g = |\boldsymbol{g}|$ and $\boldsymbol{g}$ is the acceleration of gravity), and the dissipative function Φ is described by the equality

$$\Phi = \xi({-er}\boldsymbol{V})^2 + 2D : D,$$

where D is the strain-rate tensor.

To obtain further expansions in terms of the small parameter of compressibility only, we assume that $\epsilon_1 = \alpha_1\delta$, $\epsilon_2 = \alpha_2\delta$, and $\alpha_i = O(1)$ $(i = 1, 2)$ as $\delta \to 0$. Then, Eq. (4) becomes

$$\frac{1+\delta p}{1+\varepsilon T}\,\frac{dT}{dt} - \delta\,\frac{\alpha_2+\varepsilon\alpha_1 T}{1+\varepsilon T}\,\frac{dp}{dt} = \Delta T + \delta\alpha_1\Phi. \tag{5}$$

Thus, the sought system of equations for unknown functions $\boldsymbol{V}$, p, and T are Eqs. (2), (3), and (5). Note, the equations of microconvection of an isothermally incompressible fluid are obtained from these equations under the assumption that $\delta = 0$.

Pukhnachov [5] analyzed the criteria of similarity of the problem and the characteristic parameters of the process, including the characteristic times. In constructing the convection model valid under microgravity conditions, the characteristic internal times t_* (time of temperature relaxation) and $t_\nu = l^2/\nu$ (time of relaxation of viscous stresses), which are of the same order for $\mathrm{Pr} \sim 1$, and the characteristic time t_f (time of variation of the functions determining the boundary thermal mode) are chosen. The condition $\mathrm{Pr} \sim 1$ involves a fairly large class of fluids, and the use of the relation $\zeta = t_*/t_f$ in the boundary temperature conditions allows one to consider situations where these characteristic times are strongly different.

The asymptotic expansion of the solution of system (2)–(5) is constructed in terms of the compressibility parameter $\delta \to 0$ and under the condition that ε, Pr, $\bar{\xi}$, η, α_1, and α_2 retain finite values. The solution of system (2)–(5) are sought in the form of formal power series

$$\boldsymbol{V} = \sum_{k=0}^{\infty}\delta^k\boldsymbol{V}^{(k)}(x,t), \quad T = \sum_{k=0}^{\infty}\delta^kT^{(k)}(x,t), \quad p = \frac{P(t)-1}{\delta} + \sum_{k=0}^{\infty}\delta^kp^{(k)}(x,t). \tag{6}$$

The function p has a singular component as $\delta \to 0$, and the quantity $(P(t)-1)/\delta$ is identified with the fluid pressure averaged over the domain Ω. If the walls are motionless and impermeable, the mass of the fluid contained within the cavity remains unchanged. If the total heat flux through the boundary is other than zero and, hence, the domain-averaged temperature retains a finite value, the mean pressure changes [in accordance with the equation of state (1)] by the value of the order of δ^{-1} as $\delta \to 0$. The main terms of expansions (6) satisfy the following system in the flow domain Ω:

$$\frac{P}{1+\varepsilon T^{(0)}}\,(\boldsymbol{V}_t^{(0)} + \boldsymbol{V}^{(0)}\cdot\nabla\boldsymbol{V}^{(0)}) = \mathrm{Pr}\,[\nabla\,(-p^{(0)} + \bar{\xi}{-}er\boldsymbol{V}^{(0)}) + \Delta\boldsymbol{V}^{(0)}] + \frac{\eta\mathrm{Pr}\,P}{1+\varepsilon T^{(0)}}\,\boldsymbol{g}_0; \tag{7}$$

$$\frac{\dot{P}}{P} - \frac{\varepsilon}{1+\varepsilon T^{(0)}}\,(T_t^{(0)} + \boldsymbol{V}^{(0)}\cdot\nabla T^{(0)}) + {-er}\boldsymbol{V}^{(0)} = 0; \tag{8}$$

$$\frac{P}{1+\varepsilon T^{(0)}}\,(T_t^{(0)} + \boldsymbol{V}^{(0)}\cdot\nabla T^{(0)}) - \dot{P}\,\frac{\alpha_2+\varepsilon\alpha_1T^{(0)}}{1+\varepsilon T^{(0)}} = \Delta T^{(0)} \tag{9}$$

(these equations are called the equations of convection of a weakly compressible fluid). In this case, $\dot{P} = dP(t)/dt$.

The initial-boundary problem for system (7)–(9) is formulated as follows. We consider the no-slip conditions for the velocity vector

$$\boldsymbol{V}^{(0)} = 0 \qquad (x \in \Sigma, \quad t > 0)$$

and the conditions of the second kind for temperature, which define the heat flux at the boundary of the domain Σ:

$$\frac{\partial T^{(0)}}{\partial n} = f(x, \zeta t) \qquad (x \in \Sigma, \quad t > 0). \tag{10}$$

At the initial time, we set the velocity vector and the temperature:

$$\boldsymbol{V}^{(0)} = \boldsymbol{V}_0(x), \quad T^{(0)} = T_0(x), \qquad x \in \Omega, \quad t = 0.$$

The function $P(t)$ satisfies the equation

$$\dot{P} \int\limits_{\Omega} \Big[1 - \frac{\varepsilon(\alpha_2 + \varepsilon\alpha_1 T^{(0)})}{1 + \varepsilon T^{(0)}}\Big]\, dx = \varepsilon \int\limits_{\Sigma} f \, d\Sigma$$

and the initial condition

$$P(0) = 1.$$

Note, in the limiting case with $\varepsilon = 0$, Eqs. (7)–(9) reduce to Navier–Stokes equations for an incompressible liquid.

The correctness of the formulated initial-boundary problem is examined in [5], where it is shown that the approximate solution constructed can be considered as an approximation of the order $O(\delta)$ as $\delta \to 0$ of the solution of the corresponding initial-boundary problem for the initial system (2)–(5) for $t \geqslant 1$. The formal asymptotics (6) is invalid for small times, but Eqs. (2)–(5) can be linearized near the state of isothermal equilibrium. Thus, a linear model of the transitional process arises (see [5]). The asymptotic solution of the linear problem for the transitional process does not have a pointwise limit as $\delta \to 0$, but it can be considered as the main term of internal expansion of the linearized equations of motion (2)–(5), which describe the initial stage of convection. The transitional process is accompanied by propagation of nonlinear high-frequency acoustic waves. Let us emphasize that the high-frequency acoustic oscillations were “filtered” in the resultant equations of the model of a weakly compressible fluid, and they are taken into account only at the initial stage of motion. The characteristics of oscillations and their localization are considered in [5].

The procedure of “sound filtration” was also performed in [8–11]. The model of a continuous medium, applicable for essentially subsonic flows, where the hydrodynamic approximation with “acoustic filtration” is used to describe near-critical phenomena, is considered in [12, 13]. The monograph [14] is also worth mentioning, where weakly compressible fluids are considered as fluids with a low Boussinesq number and low compressibility parameter acting as multipliers in the equation of state at temperature and pressure, respectively. Moseenkov [14] performed mathematical simulations with an asymptotically generalizing character for the classical Oberbeck–Boussinesq model and considered solvability of some axisymmetric and general three-dimensional problems, as well as some issues of solution stability.

2. Exact Solutions of Equations of Convection of a Weakly Compressible Fluid in an Infinite Band. It was noted that the equations of convection of a weakly compressible fluid admit a group with addition of an arbitrary function of time to pressure. Let us consider system (7)–(9) for the main terms of expansions of $\boldsymbol{V}^{(0)}$, $T^{(0)}$, and $p^{(0)}$ (the superscript 0 will be omitted). Let us construct the solutions of these equations, which are invariant with respect to the operator $\partial/\partial y + \varphi(t)\,\partial/\partial p$, where $\varphi(t)$ is an arbitrary function of time. The construction is performed in a manner similar to that in [1, 2].

We denote the Cartesian coordinates in space by x, y, z. Let the coordinate system be chosen so that $\boldsymbol{g}_0 = (0, -1, 0)$, and the fluid fill the layer $|x| \leqslant 1$ with the heat flux according to Eq. (10) being set at the solid boundaries of this layer. Let the heat-flux value be independent of z. The invariant solutions should have the form

$$\boldsymbol{V} = (u, v), \quad u = u(x,t), \quad v = v(x,t), \quad T = T(x,t), \quad p = \varphi(t) y + r(x,t).$$

Then, system (7)–(9) transforms to

$$\frac{\dot{P}}{P} - \frac{\varepsilon}{1 + \varepsilon T}\,(T_t + u T_x) + u_x = 0; \tag{11}$$

$$\frac{P}{1+\varepsilon T}\,(u_t + uu_x) = \mathrm{Pr}\,(-r_x + \bar{\bar{\xi}} u_{xx}); \tag{12}$$

$$\frac{P}{1+\varepsilon T}\,(v_t + uv_x) = \mathrm{Pr}\,\Big(-\varphi + v_{xx} - \frac{\eta}{1+\varepsilon T}\Big); \tag{13}$$

$$\frac{P}{1+\varepsilon T}\,(T_t + uT_x) - \dot{P}\,\frac{\alpha_2 + \varepsilon\alpha_1 T}{1+\varepsilon T} = T_{xx}. \tag{14}$$

Here $\bar{\bar{\xi}} = \bar{\xi} + 1$. We assume that the functions u, v, and $\dot{P}$ are functions of the order of the Boussinesq number ε, and the temperature T is a function of the order of unity, i.e., $u = \varepsilon U(x,t)$, $v = \varepsilon V(x,t)$, and $\dot{P} = \varepsilon f(t)$. In other words, the expansions of the functions u and v into series in powers of the small parameter ε begin from the first-order terms U and V, and the expansions of the functions T and P begin from the zeroth-order terms T^0 and 1, respectively. Then, the corollary of Eq. (11) is the relation

$$f(t) - T^0_t + U_x = 0,$$

and the corollary of the heat-transfer equation (14) is the relation

$$T^0_t = T^0_{xx}, \tag{15}$$

hence,

$$U_x = T^0_{xx} - f(t)$$

or

$$U = T^0_x - xf(t) + b(t).$$

According to the no-slip conditions, we have $U(1,t) = U(-1,t) = 0$, so we first consider the case where

$$T^0_x(-1,t) = a_-(t), \qquad T^0_x(1,t) = a_+(t), \qquad a_-(t) = a_+(t) = a(t).$$

In this case, $f = 0$, which corresponds to the condition of the zero total heat flux (see the situation described in [1, 3, 15]).

Let now $a_-(t) = -a_+(t) = a(t)$. Then, we have $b = 0$ and

$$U = T^0_x - xf(t); \tag{16}$$

$$T^0_x(-1,t) = a(t), \qquad T^0_x(1,t) = -a(t), \tag{17}$$

with, e.g., $a(t) = \mathcal{A}\sin\omega t$.

Thus, T^0 is the solution of Eq. (15) in the domain $|x| \leqslant 1$, $t \in [0, t_{\mathrm{end}}]$, with condition (17) fulfilled at the boundary, and the following initial condition can be set at the initial time:

$$T^0(x,0) = T_0(x), \qquad |x| \leqslant 1. \tag{18}$$

The function U is found from Eq. (16). Note, because of condition (17), we have $f(t) = -a(t)$. Equation (12) now determines the function $r(x,t)$ with accuracy to an arbitrary function of time:

$$r_x = -U_t/\mathrm{Pr} + \bar{\bar{\xi}} U_{xx}.$$

Equation (13) for the function $V(x,t)$ involves the function $\varphi(t)$, which is found from the condition of the zero flow rate of the fluid through an arbitrary cross section of the band $y = \mathrm{const}$ (see [1, 2]). For this purpose, we differentiate the condition $\int\limits_{-1}^{1} \rho v\,dx = 0$ with respect to t and use Eqs. (11), (13), and the equation of state $\rho = P(t)/(1+\varepsilon T)$ in the situation considered. Then, we obtain

$$\rho_t v = \frac{\varepsilon P}{(1+\varepsilon T)^2}\,T_x uv - \frac{P}{1+\varepsilon T}\,u_x v,$$

$$\rho v_t = \frac{P}{1+\varepsilon T}\,v_t = \mathrm{Pr}\Big[-\varphi + v_{xx} - \frac{\eta}{1+\varepsilon T}\Big] - \frac{P}{1+\varepsilon T}\,uv_x,$$

and, as a consequence, we determine the function

$$\varphi(t) = \frac{1}{2}\left[v_x(1,t) - v_x(-1,t)\right] - \frac{\eta}{2}\int\limits_{-1}^{1} \frac{dx}{1+\varepsilon T^0}. \tag{19}$$

Now Eq. (13) with allowance for Eq. (19) enables us to determine $V(x,t)$ as

$$V_t = -\mathrm{Pr}\,\tilde{\varphi} + \mathrm{Pr}\,V_{xx} + \mathrm{Pr}\,\eta T^0,$$

where

$$\tilde{\varphi} = \frac{1}{2}\left[V_x\right]\Big|_{-1}^{1} + \frac{\eta}{2}\int\limits_{-1}^{1} T^0\,dx,$$

and hence,

$$V_t = \mathrm{Pr}\,\Big[-\frac{1}{2}\left[V_x(1,t) - V_x(-1,t)\right] + V_{xx} + \eta T^0 - \frac{\eta}{2}\int\limits_{-1}^{1} T^0\,dx\Big]. \tag{20}$$

For this equation, we consider the following initial and boundary conditions:

$$V(x,0) = V_0(x), \qquad |x| \leqslant 1; \tag{21}$$

$$V(-1,t) = 0, \qquad V(1,t) = 0, \qquad t \in [0, t_{\mathrm{end}}]. \tag{22}$$

Each of the problems (15), (17), (18), and (20)–(22) is solved by the Fourier method. Let us consider periodic solutions of these problem, where the initial conditions are not set, the boundary conditions are described by the function $a(t) = \mathcal{A}\sin\omega t$, and the functions T^0 and V have the form

$$T^0 = T_s(x)\sin\omega t + T_c(x)\cos\omega t; \tag{23}$$

$$V = V_s(x)\sin\omega t + V_c(x)\cos\omega t. \tag{24}$$

Then, from the functions U and V, we determine the components of dimensionless velocity $u = \varepsilon U(x,t)$ and $v = \varepsilon V(x,t)$. In further comparisons with the results of the classical Oberbeck–Boussinesq model, we should bear in mind that the velocity component u in the invariant solution is constant at each time instant (and can be set equal to zero with allowance for satisfaction of the initial condition). The second velocity component v is determined by the relations written above.

2.1. *Solution of Problem* (15), (17), (18) *for Temperature.* Let us consider the equation

$$T^0_t = T^0_{xx}$$

in an infinite band $-1 \leqslant x \leqslant 1$ and consider the boundary conditions that determine the heat flux in antiphase at the band boundaries:

$$T^0_x(-1,t) = a(t), \qquad T^0_x(1,t) = -a(t).$$

Here $a(t) = \mathcal{A}\sin\omega t$. The search for a solution in the form (23) yields the following problem for T_c:

$$T_c^{(IV)} + \omega^2 T_c = 0; \tag{25}$$

$$T'_c(-1) = 0, \quad T'_c(1) = 0, \quad T'''_c(-1) = \omega\mathcal{A}, \quad T'''_c(1) = -\omega\mathcal{A}, \tag{26}$$

whereas T_s is determined in terms of T_c as

$$T_s = T''_c/\omega$$

and the following boundary conditions are satisfied for T_s:

$$T'_s(-1) = \mathcal{A}, \qquad T'_s(1) = -\mathcal{A}.$$

We introduce the notation $\vartheta = \sqrt{\omega/2}$ and $\text{æ} = \sqrt{\omega/(2\mathrm{Pr}\,)}$, where $\mathrm{Pr}\, \neq 1$. The solution of problem (25), (26) yields a linear system of algebraic equations of the form

$$-DC_1 + CC_2 - BC_3 + AC_4 = \mathcal{A}\omega/(2\vartheta^3), \qquad DC_1 + CC_2 - BC_3 - AC_4 = -\mathcal{A}\omega/(2\vartheta^3),$$

$$AC_1 + BC_2 + CC_3 + DC_4 = 0, \qquad -AC_1 + BC_2 + CC_3 - DC_4 = 0.$$

The coefficients of the system are determined as

$$A = -\sinh\vartheta\cos\vartheta + \cosh\vartheta\sin\vartheta, \qquad B = \sinh\vartheta\sin\vartheta + \cosh\vartheta\cos\vartheta,$$
$$C = \cosh\vartheta\cos\vartheta - \sinh\vartheta\sin\vartheta, \qquad D = -(\cosh\vartheta\sin\vartheta + \sinh\vartheta\cos\vartheta), \tag{27}$$

and the solution of problem (25), (26) is the function

$$T_c = C_1 \cosh\vartheta x\cos\vartheta x + C_4 \sinh\vartheta x\sin\vartheta x,$$

where

$$C_1 = \frac{\mathcal{A}\omega}{4\vartheta^3}\,\frac{\cosh\vartheta\sin\vartheta + \sinh\vartheta\cos\vartheta}{\sinh^2\vartheta\cos^2\vartheta + \cosh^2\vartheta\sin^2\vartheta}, \qquad C_4 = \frac{\mathcal{A}\omega}{4\vartheta^3}\,\frac{-\sinh\vartheta\cos\vartheta + \cosh\vartheta\sin\vartheta}{\sinh^2\vartheta\cos^2\vartheta + \cosh^2\vartheta\sin^2\vartheta}. \tag{28}$$

The function T_s has the form

$$T_s = (2\vartheta^2/\omega)[-C_1\sinh\vartheta x\sin\vartheta x + C_4\cosh\vartheta x\cos\vartheta x].$$

2.2. *Solution of Problem* (20)–(22) *for Velocity.* We consider the problem of finding a periodic solution of the form (24) for system (20)–(22), which can be rewritten in the following form for convenience:

$$V_t = \Pr V_{xx} - \frac{\Pr}{2}\,[V_x(1,t) - V_x(-1,t)] - \frac{\Pr\eta}{2}\int\limits_{-1}^{1} T^0\,dx + \Pr\eta T^0,$$

$$V(-1,t) = 0, \qquad V(1,t) = 0, \qquad t \in [0, t_{\mathrm{end}}].$$

The function V_c is found by solving the inhomogeneous ordinary differential equation

$$V_c^{(IV)} + \frac{\omega^2}{\Pr^2}\,V_c = \frac{1}{2}\,[V_c'''(1) - V_c'''(-1)] - \eta T_c'' + \frac{\eta\omega}{\Pr}\,T_s + \frac{\eta\omega}{2\Pr}\,I_1.$$

The function V_s is determined from the relation

$$V_s = \frac{\Pr}{\omega}\,V_c'' - \frac{\Pr}{2\omega}\,[V_c'(1) - V_c'(-1)] - \frac{\Pr\eta}{2\omega}\,I_2 + \frac{\Pr\eta}{\omega}\,T_c. \tag{29}$$

Here

$$I_1 = \int\limits_{-1}^{1} T_s(x)\,dx, \qquad I_2 = \int\limits_{-1}^{1} T_c(x)\,dx.$$

The solution of Eq. (29) is constructed as the sum of the general solution of the homogeneous equation and the partial solution determined by the right side of Eq. (29):

$$V_c = \bar{C}_1\cosh æx\cos æx + \bar{C}_2\cosh æx\sin æx + \bar{C}_3\sinh æx\cos æx + \bar{C}_4\sinh æx\sin æx + \tilde{V},$$

$$\tilde{V} = G_0 + G_1\sinh\vartheta x\sin\vartheta x + G_4\cosh\vartheta x\cos\vartheta x.$$

With allowance for the boundary conditions

$$V_c(-1) = 0, \qquad V_c(1) = 0, \qquad V_s(-1) = 0, \qquad V_s(1) = 0$$

the coefficients $\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3$, and $\bar{C}_4$ are found as the solutions of the linear algebraic system

$$\tilde{A}\bar{C}_1 + \tilde{B}\bar{C}_2 + \tilde{C}\bar{C}_3 + \tilde{D}\bar{C}_4 = \tilde{E}, \qquad \tilde{A}\bar{C}_1 - \tilde{B}\bar{C}_2 - \tilde{C}\bar{C}_3 + \tilde{D}\bar{C}_4 = \tilde{E},$$
$$K\bar{C}_1 + L\bar{C}_2 + M\bar{C}_3 + N\bar{C}_4 = F, \qquad K\bar{C}_1 - L\bar{C}_2 - M\bar{C}_3 + N\bar{C}_4 = F. \tag{30}$$

The coefficients of this system are determined as follows:

$$\tilde{A} = \bar{A} + \bar{\Phi}_1, \quad \bar{A} = \cosh æ\cos æ, \qquad \tilde{D} = \bar{D} + \bar{\Phi}_4, \quad \bar{D} = \sinh æ\sin æ,$$

$$\tilde{B} = \bar{B} = \cosh\,æ\sin æ, \quad \tilde{C} = \bar{C} = \sinh\,æ\cos æ, \quad \tilde{E} = -(\Phi_0 + G_1\sinh\,\vartheta\sin\vartheta + G_4\cosh\,\vartheta\cos\vartheta),$$

$$K = -2æ^2\bar{D}\,\frac{\mathrm{Pr}}{\omega} - 2æ(\bar{C}-\bar{B})\,\frac{\mathrm{Pr}}{2\omega}, \qquad L = 2æ^2\bar{C}\,\frac{\mathrm{Pr}}{\omega},$$

$$M = -2æ^2\bar{B}\,\frac{\mathrm{Pr}}{\omega}, \qquad N = 2æ^2\bar{A}\,\frac{\mathrm{Pr}}{\omega} - 2æ(\bar{C}+\bar{B})\,\frac{\mathrm{Pr}}{2\omega},$$

$$F = \frac{\mathrm{Pr}\,\eta}{2\omega}\,I_2 - \frac{\mathrm{Pr}\,\eta}{\omega}\,(C_1\bar{\Phi}_c + C_4\bar{\Phi}_s) - \frac{\mathrm{Pr}}{2\omega}\,2\vartheta(G_1 D + G_4 A) - \frac{\mathrm{Pr}}{\omega}\,2\vartheta^2(G_1\bar{\Phi}_c - G_4\bar{\Phi}_s).$$

The coefficients A and D are calculated by formulas (27), and the coefficients C_1 and C_4 are calculated by formulas (28). Here we calculate

$$I_1 = \frac{2\vartheta}{\omega}\,(C_4 - C_1)\cosh\,\vartheta\sin\vartheta + \frac{2\vartheta}{\omega}\,(C_4 + C_1)\sinh\,\vartheta\cos\vartheta,$$

$$I_2 = \frac{1}{\vartheta}\,(C_1 + C_4)\cosh\,\vartheta\sin\vartheta + \frac{1}{\vartheta}\,(C_1 - C_4)\sinh\,\vartheta\cos\vartheta,$$

and introduce the following notation for convenience:

$$\Phi_0 = \frac{\eta\omega}{8\mathrm{Pr}\,æ^4}\,I_1, \quad \bar{\Phi}_1 = \frac{\Phi_1}{4æ^4}, \quad \bar{\Phi}_4 = \frac{\Phi_4}{4æ^4}, \quad \bar{\Phi}_c = \cosh\,\vartheta\cos\vartheta, \quad \bar{\Phi}_s = \sinh\,\vartheta\sin\vartheta,$$

$$\Phi_1 = -4æ^3(\cosh\,æ\sin æ + \sinh\,æ\cos æ)/2, \qquad \Phi_4 = -4æ^3(\sinh\,æ\cos æ - \cosh\,æ\sin æ)/2,$$

$$G_0 = \Phi_0 + \bar{\Phi}_1\bar{C}_1 + \bar{\Phi}_4\bar{C}_4, \qquad G_1 = F_1/(4(æ^4 - \vartheta^4)), \qquad G_4 = F_4/(4(æ^4 - \vartheta^4)),$$

$$F_1 = 2\eta\vartheta^2 C_1 - \frac{\eta\omega}{\mathrm{Pr}}\,\frac{2\vartheta^2}{\omega}\,C_1, \qquad F_4 = -2\eta\vartheta^2 C_4 + \frac{\eta\omega}{\mathrm{Pr}}\,\frac{2\vartheta^2}{\omega}\,C_4.$$

The solution of system (30) has the form

$$\bar{C}_1 = \bar{\Delta}_1/\bar{\Delta}, \qquad \bar{C}_4 = \bar{\Delta}_4/\bar{\Delta}, \qquad \bar{C}_2 = \bar{C}_3 = 0,$$

where the denominator is determined as

$$\bar{\Delta} = (\bar{A} + \bar{\Phi}_1)\Big(2æ^2\bar{A}\,\frac{\mathrm{Pr}}{\omega} - 2æ(\bar{C}+\bar{B})\,\frac{\mathrm{Pr}}{2\omega}\Big) - \Big(-2æ^2\bar{D}\,\frac{\mathrm{Pr}}{\omega} - 2æ(\bar{C}-\bar{B})\,\frac{\mathrm{Pr}}{2\omega}\Big)(\bar{D} + \bar{\Phi}_4),$$

and the numerators are written as

$$\bar{\Delta}_1 = -(\Phi_0 + G_1\sinh\,\vartheta\sin\vartheta + G_4\cosh\,\vartheta\cos\vartheta)\Big(2æ^2\bar{A}\,\frac{\mathrm{Pr}}{\omega} - 2æ(\bar{C}+\bar{B})\,\frac{\mathrm{Pr}}{2\omega}\Big)$$

$$-\Big(\frac{\mathrm{Pr}\,\eta}{2\omega}\,I_2 - \frac{\mathrm{Pr}\,\eta}{\omega}\,(C_1\bar{\Phi}_c + C_4\bar{\Phi}_s) - \frac{\mathrm{Pr}}{2\omega}\,2\vartheta(G_1 D + G_4 A) - \frac{\mathrm{Pr}}{\omega}\,2\vartheta^2(G_1\bar{\Phi}_c - G_4\bar{\Phi}_s)\Big)(\bar{D} + \bar{\Phi}_4),$$

$$\bar{\Delta}_4 = (\bar{A} + \bar{\Phi}_1)\Big(\frac{\mathrm{Pr}\,\eta}{2\omega}\,I_2 - \frac{\mathrm{Pr}\,\eta}{\omega}\,(C_1\bar{\Phi}_c + C_4\bar{\Phi}_s) - \frac{\mathrm{Pr}}{2\omega}\,2\vartheta(G_1 D + G_4 A) - \frac{\mathrm{Pr}}{\omega}\,2\vartheta^2(G_1\bar{\Phi}_c - G_4\bar{\Phi}_s)\Big)$$

$$-\Big(2æ^2\bar{D}\,\frac{\mathrm{Pr}}{\omega} + 2æ(\bar{C}-\bar{B})\,\frac{\mathrm{Pr}}{2\omega}\Big)(\Phi_0 + G_1\sinh\,\vartheta\sin\vartheta + G_4\cosh\,\vartheta\cos\vartheta),$$

and hence, we have

$$V_c(x) = \bar{C}_1\cosh\,æx\cos æx + \bar{C}_4\sinh\,æx\sin æx + \tilde{V}(x),$$

$$\tilde{V}(x) = G_0 + G_1\sinh\,\vartheta x\sin\vartheta x + G_4\cosh\,\vartheta x\cos\vartheta x.$$

The following relation is valid for the function V_s:

$$V_s(x) = \mathrm{Pr}\ [-2\bar{C}_1æ^2\sinh\,æx\sin æx + 2\bar{C}_4æ^2\cosh\,æx\cos æx$$

$$+\,2\vartheta^2(G_1\cosh\,\vartheta x\cos\vartheta x - G_4\sinh\,\vartheta x\sin\vartheta x)]/\omega$$

TABLE 1

Calculation variant	Pr	η	ε	ν, cm^2/sec	χ, cm^2/sec	g, cm/sec^2	β, deg^{-1}	ω, sec^{-1}
1	0.75	1.0	0.01; 0.5	0.150	0.2	0.030	0.0003	0.5; 2.5; 5
2	0.01	0.4	0.01; 0.5	0.015	1.5	0.009	0.0006	0.5; 2.5; 5
3	0.10	0.4	0.02; 0.5	0.150	1.5	0.090	0.0006	0.5; 5

$$-\,\mathrm{Pr}\,[2\text{æ}\bar{C}_1(\bar{C}-\bar{B})+2\text{æ}\bar{C}_4(\bar{C}+\bar{B})-2\vartheta(G_1D+G_4A)]/(2\omega)-\mathrm{Pr}\,\eta I_2/(2\omega)$$

$$+\mathrm{Pr}\,\eta[C_1\cosh\vartheta x\cos\vartheta x+C_4\sinh\vartheta x\sin\vartheta x]/\omega.$$

Thus, we determine the functions V_c and V_s, and simultaneously, $V(x,t)$ of the form (24).

Note, in real situations, the Boussinesq numbers ε are small. The present analysis of the linearized problem is fairly justified because its solution is given by the main term of the asymptotic solution as $\varepsilon \to 0$ (see a similar justification for the microconvection model in [15]).

3. Trajectory Calculations. The components of the physical (dimensional) velocity are determined as $v_1 = v_*u$ and $v_2 = v_*v$, where $u = \varepsilon U$, $v = \varepsilon V$, and $v_* = \chi/l$, and formulas (16) and (24) are used for U and V. Knowing the functions v_1 and v_2, we can calculate the trajectories of fluid particles. For this purpose, we have to solve the Cauchy problem

$$\frac{dx}{dt} = v_1(x,t), \quad \frac{dy}{dt} = v_2(x,t), \quad t > 0, \qquad x(0) = x_0, \quad y(0) = 0. \tag{31}$$

Note, in constructing trajectories of fluid particles by the Oberbeck–Boussinesq model, we have to use $v_1 = 0$, whereas the expression for v_2 remains unchanged.

The objective of the present work was to determine the trajectories of fluid particles from the calculation results based on the model of convection of a weakly compressible fluid. The differences from the results predicted by the classical convection model allow us to conclude that there are non-Boussinesq effects and justify the use of new mathematical models of convection. In addition, the problem of comparing the results with the data obtained by the model of microconvection of an isothermally incompressible fluid is posed. Formulation of the initial-boundary problem for microconvection equations implies setting the boundary heat flux under the condition that the integral heat flux equals zero. In the problem of convection of a heat-conducting fluid in an infinite band, this is manifested in the phase changes in the boundary thermal mode, i.e., one lateral boundary is heated, and the other lateral boundary is simultaneously cooled. It became possible to consider the boundary heat flux periodic in time and changing in antiphase in simulating convection under microgravity conditions owing to the new mathematical model of convection of a weakly compressible fluid.

Projections of the integral curves of system (31) onto the plane (x,y), which were calculated by the microconvection model with the parameters $\varepsilon = 0.01$ and 0.02, $\omega = 0.5$ and 2 sec^{-1}, are given in [1, 2] and demonstrate the helical (the main coil is an ellipse) periodic motion of the fluid particle. As was noted in [1], it is rather difficult to analyze the behavior of the trajectories because of the variety of dimensionless parameters that affect the solution of the Cauchy problem (31). Nevertheless, we can assume that, under conditions of applicability of the microconvection model and with the use of the model of convection of a weakly compressible fluid under the same conditions, the intensity of periodic motion and the particle drift are primarily determined by the values of the angular frequency ω, the Boussinesq parameter ε, and naturally, by the position of the point (x_0, y_0) relative to the lateral boundaries of the domain. These assumptions were validated in [15].

The main parameters of the problem are listed in Table 1 and are conventionally presented by three models of fluid media and physical situations with different values of Pr, η, and g, similarly to that considered in [15]. The characteristic velocities, Reynolds numbers, and times of the process are also different. To demonstrate the behavior of the trajectories, whose time evolution is rather complicated, we choose the values $\varepsilon = 0.5$ and 0.02, $\omega = 0.5$ and 5 sec^{-1}. The value $\varepsilon = 0.5$ is chosen to demonstrate the dependence on the Boussinesq number, which is most important for trajectory development, and to obtain illustrative results.

The calculations were performed for $\mathcal{A} = -1$ [see the boundary condition (17)], which implies heating of the right boundary $x = 1$ both in the microconvection model and in the model of a weakly compressible fluid. This allows simple comparisons with the results described in [1, 2, 15]. Figure 1a–c for variants 1–3, respectively, shows the trajectories of fluid particles calculated by three convection models. The trajectories calculated by the classical

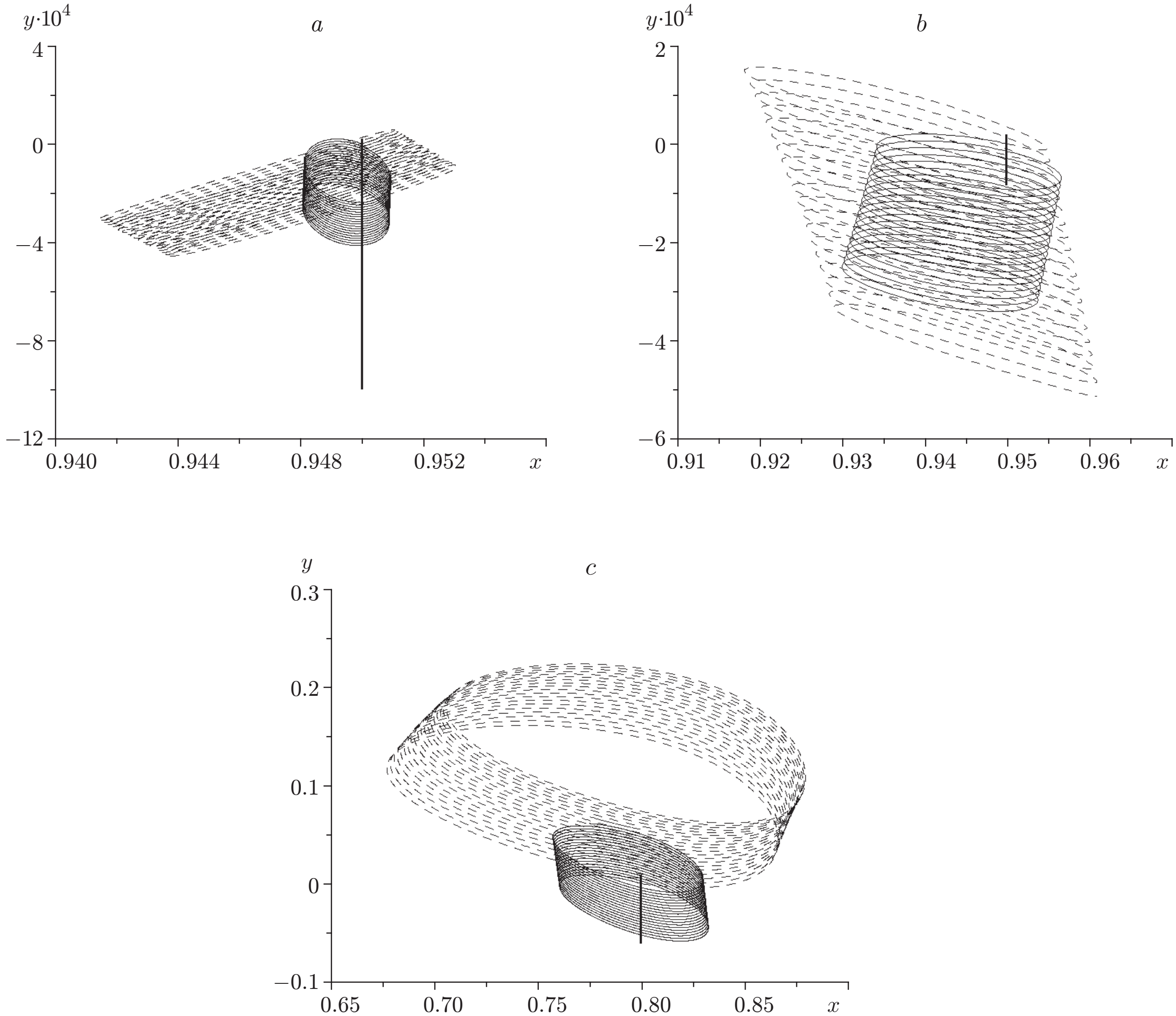

Fig. 1. Trajectory of the fluid particle ($\varepsilon = 0.5$, $\omega = 5$ sec^{-1}, and $\mathcal{A} = -1$) for variants 1 (a) and 2 (b) ($t = 0$–24 sec, $x_0 = 0.95$, and $y_0 = 0$) and variant 3 (c) ($t = 0$–240 sec, $x_0 = 0.8$, and $y_0 = 0$).

Oberbeck–Boussinesq model are shown as vertical segments of straight lines, and the trajectories calculated by the microconvection model display helical motion. They are marked by the dashed curves. The trajectories calculated by the model of convection of a weakly compressible fluid are also of the helical type and are marked by the solid curves. Figures 1a and 1b for $\varepsilon = 0.5$ and $\omega = 5$ sec^{-1} show the trajectories in the time interval from 0 to 24 sec, for the fluid particle located at the initial time $t = 0$ at the point $x_0 = 0.95$, $y_0 = 0$. Figure 1c for $\varepsilon = 0.5$ and $\omega = 0.5$ sec^{-1} shows the trajectories of the fluid particle in the time interval from 0 to 240 sec (at the initial time $t = 0$, the particle is located at the point $x_0 = 0.8$, $y_0 = 0$). The Oberbeck–Boussinesq model describes the motion over the vertical segment of the straight line $x = 0.95$ (Fig. 1a and b) or $x = 0.8$ (Fig. 1c).

Figure 2 shows the calculation results for variant 2 for $\varepsilon = 0.5$ and $\omega = 2.5$ sec^{-1} the fluid particle at the initial time is located at the point $(0.95, 0)$. The particle drift is tracked in the time interval from 0 to 600 sec. This figure shows the complicated helical motion in accordance with the model of convection of a weakly compressible fluid. Figure 2b shows the particle trajectory for $\mathcal{A} = 1$ [see condition (17)], which corresponds to cooling of the right boundary $x = 1$. For comparison with the microconvection model, we should say that the particle-drift direction changes in the model of a weakly compressible fluid and remains unchanged in the microconvection model (see Fig. 2a: the downward motion is replaced by the upward motion).

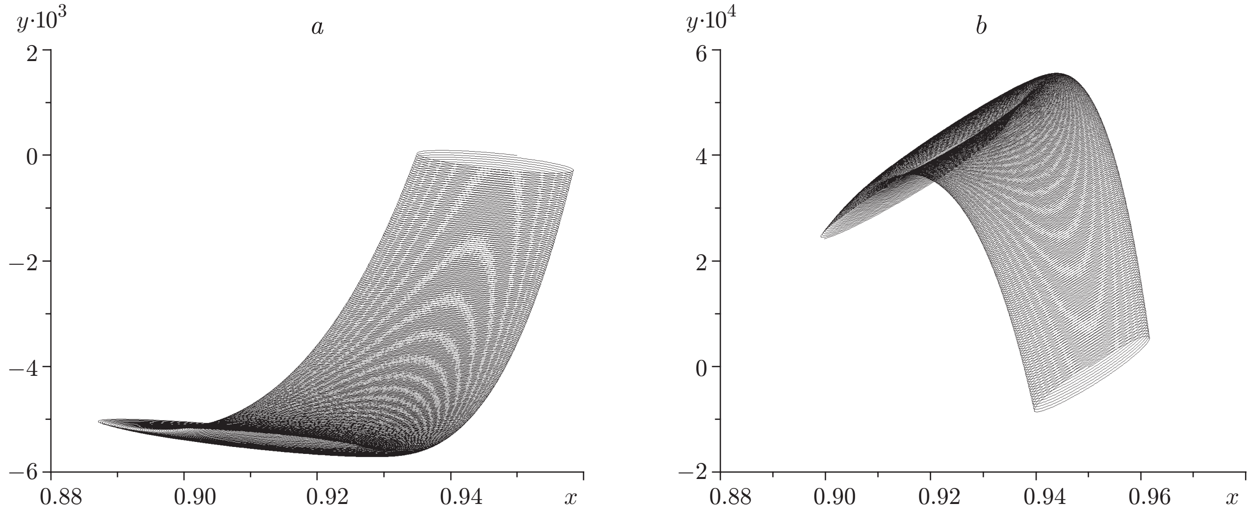

Fig. 2. Trajectory of the fluid particle for variant 2 with $t = 0$–600 sec, $x_0 = 0.95$, $y_0 = 0$, $\varepsilon = 0.5$, $\omega = 5\ \text{sec}^{-1}$, and $\mathcal{A} = -1$ (a) and 1 (b).

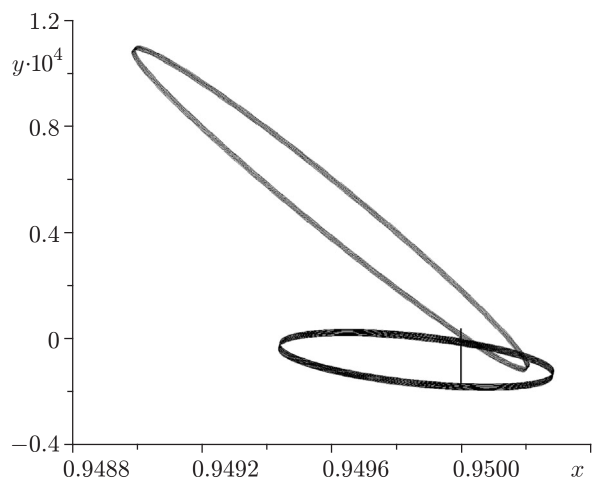

Fig. 3. Trajectory of the fluid particle for variant 3 with $t = 0$–24 sec, $x_0 = 0.95$, $y_0 = 0$, $\varepsilon = 0.02$, $\omega = 5\ \text{sec}^{-1}$, and $\mathcal{A} = -1$.

Figure 3 shows the trajectories calculated for variant 3 in the time interval from 0 to 24 sec for $\varepsilon = 0.02$ and $\omega = 5\ \text{sec}^{-1}$. At the initial time, the fluid particle is located at the point $(0.95, 0)$. By comparing the trajectories, we can note that the microconvection model predicts a helical trajectory with a larger diameter than that in calculations by the model of a weakly compressible fluid.

All calculations were performed until the final time $t_{\text{end}} = 2400$ sec. The Cauchy problem for the system of ordinary differential equations (31) was numerically examined by the Runge–Kutta method [16].

The author is sincerely grateful to V. V. Pukhnachov for discussing the problem formulation and the results obtained.

This work was supported by the Foundation "Leading Scientific Schools of Russia" (Grant No. NSh 902.2003.01).

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